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On the Solvability of Certain Factorizable Groups, II

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We continue the investigation, begun in [3], of sufficient conditions on the factors A and B of a finite factorizable group $G = AB$ to imply the solvability of G. A finite group is called Dedekind if all its subgroups are normal. The structure of Dedekind groups is well known ([6], p. 308); in particular, a Dedekind group of odd order is necessairly Abelian. We prove

MAIN THEOREM. *If the finite group G has subgroups A and B so that $G = AB$ and A has a nilpotent subgroup of index 2 and B is Dedekind then G is solvable.*

This generalizes results that the finite group G is solvable if it is the product of two subgroups A and B where A has a nilpotent subgroup index 2 and B is cyclic (Huppert and Itô [7] and Knop [10]) or if A has a Dedekind subgroup of index 2 and B is Dedekind (Knop [10]).

We quote some results which we shall need below. Many of these results can be found in [13], Chapter 13. Our notations are standard (see [6]). In particular, $H \leqslant G$ ($H < G$) means H is a subgroup (proper subgroup) of G, and $O_p(G)$ is the largest normal p-subgroup of G, for p a prime. Throughout, G is a finite group and A and B subgroups so that $G = AB$.

THEOREM A. *If A and B have non-trivial centers, and $C_G(b) \leqslant B$ for every non-identity element $b \in B$, then G is not simple.* (Itô, [8]).

THEOREM B. *If A_1 and B_1 are normal subgroups of A and B, respectively, and if A_1B_1 is a proper subgroup of G, then G is not simple. (Hilfsatz 10 of [15] and Satz 3 of [9]).*

THEOREM C. *If p is a prime dividing both A and B, and if A and B have normal Sylow p-subgroups, and if every proper quotient group of G and every proper subgroup of G containing A or B is solvable, then G is solvable. (Essentially Satz 1 of [9]; proved in this form in [3]).*

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THEOREM D. *If A has a nilpotent subgroup H of index 2, and B is nilpotent, and either H or B is a p-group for some prime p, then G is solvable.* [3]

THEOREM E. *If A_1 and B_1 are normal subgroups of A and B, respectively, and $L = N_G(\langle A_1, B_1\rangle)$, then $L = (L \cap A)(L \cap B)$. (Theorem 13.2.7 of [13]).*

THEOREM F. *If G is solvable and A_1 and B_1 are normal Hall subgroups of A and B, respectively, then A_1B_1 is a subgroup of G.* [15]

THEOREM G. *If some non-identity subgroup of A is normal in B, then A contains a non-identity normal subgroup of G. (Many authors, including [10]).*

THEOREM H. *If C is any finite group, and H is a nilpotent Hall subgroup of C and if every Sylow subgroup of H has normalizer H, then H has a normal complement in C. (Hilfssatz 9 of [15]).*

We also need the following lemma, which has been used in various forms by a number of authors, including Scott in [12] and Monahov in [11].

LEMMA 1. *If $G = AB$, where $A \cap B = 1$, B is a proper subgroup of G and $|A|$ is twice an odd number, then G has a subgroup K of index 2 such that no involution of A is contained in K.*

Proof. Let G act on the set $\mathscr{S}$ of right cosets of B in G by right multiplication. This action is faithful; if it is not faithful, the kernel of the action is a non-trivial normal subgroup N of G contained in B. Then G/N satisfies the hypotheses of the lemma, and so by induction G/N, and therefore G, has a subgroup as required. Also, A acts regularly on $\mathscr{S}$. Therefore any involution of A is represented as the product of an odd number of disjoint transpositions, and so as an odd permutation. Thus the elements of G represented as even permutations form a subgroup of index 2 as required.

In the proof of Theorem 1, we need the following lemma in a crucial place.

LEMMA 2. *Let $G = AB$, with $|A|$ and $|B|$ relatively prime, A a maximal subgroup of G, B Abelian, H a nilpotent subgroup of A of index 2, and $|H| = 2^\alpha p^\beta$, for p an odd prime, $\beta \neq 0$. Assume further that H_p, the Sylow p-subgroup of H, is non-Abelian, and that a proper subgroup K of G is solvable if $K = (K \cap A)(K \cap B^g)$ for some $g \in G$. Then G is not simple.*

Proof. Suppose the result is false, and let G be such a simple group. Then $2 \mid |H|$, or else $|G|_2 = 2$, and G would not be simple. Let H_2 be the Sylow 2-subgroup of H and A_2 a Sylow 2-subgroup of A. By the maximality of A, we have $N_G(H_2) = N_G(H_p) = A$.

We show

(1) if X is any proper subgroup of G with $H_p \leqslant X$, then X is a $\{2, p\}$-subgroup. If such an X contains an element of order dividing $|B|$, then X intersects non-trivially some conjugate of B, say $X \cap B^g > 1$; we call $E = \langle H_p, X \cap B^g \rangle$. Then since $G = AB^g$ and by Theorem E, $N_G(E)$ factorizes. Since $E \leqslant X$, E is a proper subgroup of G and so is $N_G(E)$, and then by the hypothesis on proper factorizable subgroups, $N_G(E)$ is solvable. Now it is easy to see that in a solvable group with a factorization by subgroups of relatively prime order, a normal subgroup also factorizes, and so $E = (E \cap A) \cdot (E \cap B^g)$. But then by Theorem F applied to the group E, $H_p \cdot (E \cap B^g)$ is a proper subgroup of G, which by Theorem B contradicts the simplicity of G. Thus (1) is established.

We now show

(2) if X is a subgroup of G and $H_p \leqslant X$, then $X \leqslant A$. First, by (1) X is solvable. If $O_p(X) \neq 1$, set $U = O_p(X)$. Then $U \lhd H_p$, so $H \leqslant N_G(U)$. Now by (1), $N_G(U)$ is a $\{2, p\}$-group, so $|N_G(U): H| = 1$ or 2. Thus H is normal in $N_G(U)$, so $N_G(U) \leqslant N_G(H) = A$, and so $X \leqslant A$.

So we may assume $O_p(X) = 1$, and we let $V = O_2(X) > 1$.

We now claim that p divides $|C_G(V)|$. Take $c \in G$ so that $V \leqslant A_2{}^c$. If $V \leqslant H_2{}^c$, then $H_p{}^c \leqslant C_G(V)$, and the claim is established. If $V \nleqslant H_2{}^c$, then any non-identity element $x \in V - V \cap H_2$ induces a non-trivial automorphism of $H_p{}^c$ of order 2, since $x^2 \in H_2{}^c$. This automorphism is non-trivial, or else A^c is nil-potent and G is solvable by the Kegel-Wielandt Theorem, and so, since $H_p{}^c$ is non-Abelian, this automorphism has non-identity fixed points, which therefore belong to $C_G(V)$, establishing the claim that p divides $|C_G(V)|$. Since X contains a full Sylow p-subgroup of G, and $X \leqslant N_G(V)$ and $C_G(V) \trianglelefteq N_G(V)$ we calculate

$$C_X(V) = \frac{|X| \cdot |C_G(V)|}{|XC_G(V)|}$$

and see that $p \mid |C_X(V)|$, also.

Since we have $O_{2'}(X) = 1$ here, we may apply the Lemma of Hall and Higman ([6], Hilfsatz VI. 6.5) to obtain $C_X(V) \leqslant V$, contrary to $p \mid |C_X(V)|$. This contradiction shows $X \leqslant A$.

We now show

(3) H_2 is tightly embedded in G; in fact, we show that if $x \notin N_G(H_2)$ and $x \neq 1$, then $H_2 \cap H_2{}^x = 1$. Obviously it suffices to show this with $x = b \in B$. Now if $1 \neq y \in H_2 \cap H_2{}^b$, then $H_p, H_p{}^b \leqslant C_G(y) \leqslant A$, this last by (2). But H_p is the unique Sylow p-subgroup of A, so $H_p = H_p{}^b$, contrary to $N_G(H_p) = A$, establishing (3).

Now a theorem of Aschbacher (Theorem 2 of [2]) applies to this situation. For $b \in B$, let $T = H_2{}^b \cap A$ (which, by (3), has order 1 or 2) and let $S = H_2 T$. By Aschbacher's result, since H_2 is tightly embedded in G, either

(i) for some $b \in B$, $T \neq 1$ and $N_S(T) \cong T \times T$, or

(ii) $T = 1$ for every $b \in B$.

Now (ii) cannot occur, for let $x \in H_2 \cap Z(A_2)$, $x \neq 1$. If $x^b \notin A_2$, for all $b \in B$, then by Glauberman's Z^*-theorem ([4], Cor. 1), $x \in Z(G)$, contrary to the simplicity of G. Thus for some $b_0 \in B$, $x^{b_0} \in A_2$, and so $H^{b_0} \cap A \neq 1$.

Thus (i) holds, and $|T| = 2$, and so S is a Sylow 2-subgroup of G and $N_S(T) \cong T \times T$.

Now a 2-group with an involution whose normalizer has order 4 must be dihedral or semi-dihedral ([14], Lemma 4). But simple groups with such Sylow 2-subgroups have been classified ([1] and [5]) and in these simple groups, centralizers of involutions have Abelian cores (that is, the largest normal subgroup of odd order of $C_G(x)$ is Abelian for x an involution), whereas in our case an involution in $H_2 \cap Z(A_2)$ has A as its centralizer and the non-Abelian H_p in the core. This contradiction completes the proof of the lemma.

We now prove the Main Theorem by examining a number of special cases (Theorems 1–3) depending on whether $|H|$ and $|B|$ are even or odd.

THEOREM 1. *If $G = AB$ satisfies the hypotheses of the Main Theorem, and $|B|$ is odd, then G is solvable.*

Proof. Since B is Dedekind and of odd order, we are assuming here that B is Abelian. Suppose the theorem is false, and let G be a minimal counterexample. If N is a non-identity normal subgroup of G, then $G/N = (AN/N) \cdot (BN/N)$ and AN/N contains a nilpotent subgroup, HN/N, of index 1 or 2 and BN/N is Abelian of odd order, so G is solvable by the Kegel-Wielandt Theorem or by induction. Similarly, using the Dedikind identity, every proper subgroup of G containing A or B factorizes and so is solvable.

Since every Sylow p-subgroup of H is normal in A, and for $p \neq 2$ is a Sylow subgroup of A, $|A|$ and $|B|$ are relatively prime, or G would be solvable by Theorem C. From this it follows that G is simple, for let N be any non-identity normal subgroup of G, and consider AN. If AN is proper in G, then AN is solvable as $A \leqslant AN$ and so N is solvable. Since G/N is also solvable, this implies G is solvable. Thus $AN = G$, and similarly $BN = G$. As $(|A|, |B|) = 1$, this implies $G = N$, and so G is simple.

Furthermore A is a maximal subgroup of G. If not let $A < M < G$. Then $G = M \cdot B$ and $1 < M \cap B \trianglelefteq B$, since B is Abelian. Then Theorem G shows that G is not simple, a contradiction.

We can now apply Itô's Theorem (Theorem A). Let H_2 be the Sylow 2-subgroup of H, and A_2 a Sylow 2-subgroup of A. Then $1 < H_2 \cap Z(A_2) \leqslant Z(A)$, so $Z(A)$ is non-trivial. Obviously $Z(B) = B$ is non-trivial, and so by the simplicity of G and Itô's Theorem, there exists a non-identity element $b_0 \in B$ so that $C_G(b_0) \nleqslant B$. Thus since B is Abelian, $B < C_G(b_0)$, and so by the

Dedekind Identity, $C_G(b_0) = B \cdot (C_G(b_0) \cap A)$, where $C_G(b_0) \cap A > 1$. Call $C_G(b_0) \cap A = A_1$. We now claim:

$$A_1 \cap H = 1. \tag{4}$$

Suppose to the contrary that $A_1 \cap H > 1$, and let $y \in A_1 \cap H$, $y \neq 1$, y of prime power order. Thus for some prime p, $y \in H_p$, the Sylow p-subgroup of H.

We now show that there exists a subgroup F which is a normal Sylow subgroup of A which commutes with y.

If $p = 2$, we take $F = H_q$, where H_q is the Sylow q-subgroup of H for any odd prime q; such exist or A would be a 2-group and G would be solvable by the Kegel-Wielandt Theorem. If $p \neq 2$ and H_p happens to be Abelian, we take $F = H_p$. If $p \neq 2$ and if there is another prime $q \neq 2, p$, which divides $|H|$, we take $F = H_q$. If none of these cases occured, then H would be a $\{2, p\}$-group with H_p non-Abelian, and so the hypotheses of Lemma 2 would hold, and G would be non-simple, a contradiction. Thus we may find an F as required.

Thus $\langle F, b_0 \rangle \leqslant C_G(y) < G$, this last since G is simple. Now as in the proof of Lemma 2, $L = N_G(\langle F, b_0 \rangle)$ is a proper factorizable group, and is solvable, and so $F \cdot (L \cap B)$ is a subgroup by Theorem F. But this implies, by Theorem B, that G is not simple, a contradiction. This contradiction establishes (4) and so $A_1 \cap H = 1$ and so $|A_1| = 2$.

Therefore, by counting orders, we see that $G = H \cdot C_G(b_0)$ with $B \leqslant C_G(b_0)$, $[C_G(b_0) \colon B] = 2$, and so $|C_G(b_0)|$ is twice an odd number. Therefore, by Lemma 1, G has a normal subgroup of index 2, contrary to the simplicity of G. This contradiction completes the proof of Theorem 1.

Before continuing with the proof of the Main Theorem, we give the following easy corollary of Theorem 1.

COROLLARY. *If $G = AB$, A has a subgroup H of index 2 where H is Abelian of odd order and B is nilpotent, then G is solvable.*

Proof. Suppose not, and let G be a minimal counter-example. As in the theorem, any proper quotient group of G and any proper subgroup of G containing A or B is solvable.

If $2 \nmid |B|$, then $|G|_2 = 2$, and so G has a normal 2-complement C. Then $B \leqslant C$, so C is solvable, and G/C is solvable, so G is solvable. Thus we may assume that $2 \mid |B|$.

Now let B_2 be the Sylow 2-subgroup of B, and let $M = N_G(B_2)$. Since $B \leqslant M$, M is solvable, and so we may take F a Hall $\pi(B)$-subgroup of M containing B. Then by counting orders, we have $G = HF$, $[F : B] = 2$, and so G is solvable by Theorem 1, and the corollary is proved.

We now continue with the proof of the Main Theorem.

THEOREM 2. *If $G = AB$ satisfies the hypotheses of the Main Theorem, and $|H|$ is odd, then G is solvable.*

Proof. Suppose the theorem is false, and let G be a minimal counter-example. As in Theorem 1, every proper quotient group of G and every proper subgroup of G containing A or B is solvable, and by Theorem C, $|H|$ and $|B|$ are relatively prime, and by Theorem G we have $A \cap B = 1$. Furthermore A is a maximal subgroup of G, for if there is a subgroup M satisfying $A < M < G$ then M is solvable, and by Theorem G, we see that M contains a solvable normal subgroup of G, making G solvable. The maximality of A implies that for any Sylow p-subgroup H_p of H, we have $N_G(H_p) = A$.

From Lemma 1, since $|A|$ is twice an odd number, G contains a normal subgroup N of index 2. Now if L is any proper normal subgroup of G, then as in Theorem 1, $AL = G$, and $BL = G$. Since $(|A|, |B|) = 2$, this implies $|G:L| = 2$. Then if $L \neq N$, $L \cap N = 1$, since any proper normal subgroup has order equal to $|N|$, and so we conclude that $L = N$. Thus N is the only proper normal subgroup of G, making N characteristic-simple, and therefore the direct product of isomorphic simple groups.

Now as H has odd order, $H \leqslant N$. If $A \cap N = A$, then $N = A \cdot (B \cap N)$ and by induction N, and therefore G, is solvable. Thus $A \cap N = H$.

We can now apply a result of Wielandt (Theorem H) to the subgroup N. By Theorem D we may assume that H is not a Sylow subgroup of G. Since $(|H|, |B|) = 1$, H is a Hall subgroup of N. And for any Sylow p-subgroup H_p of H, we have

$$N_N(H_p) = N \cap N_G(H_p) = N \cap A = H.$$

Thus by Theorem H, there is a subgroup K, normal in N, so that $N = HK$ and $H \cap K = 1$. But N is the direct product of isomorphic simple groups, and N/K is nilpotent, which implies that these simple groups are actually cyclic. Thus N and therefore G is solvable. This contradiction completes the proof of Theorem 2.

The following theorem, together with Theorems 1 and 2, completes the proof of the Main Theorem.

THEOREM 3. *Let $G = AB$ satisfy the hypotheses of the Main Theorem, and suppose $|H|$ and $|B|$ are both even. Then G is solvable.*

Proof. Suppose the theorem is false, and let G be a counter-example of minimal order. By the Kegel-Wielandt Theorem, or by Theorem 1 or Theorem 2, or by induction, every proper quotient group of G and every proper subgroup of G containing A or B is solvable. As in Theorem 2 we have $A \cap B = 1$ and A is a maximal subgroup of G, and by Theorem C, $(|A|, |B|)$ is a power of 2.

Now there exist non-trivial subgroups H_2, A_2, B_2 and G_2, Sylow 2-subgroups of H, A, B and G respectively, so that $G_2 = A_2B_2$ and $H_2 \leqslant A_2$.

We claim $|H_2| = 2$. Since $A_2 < N_{G_2}(A_2)$, we have $N_{G_2}(A_2) = A_2 \cdot (N_{G_2}(A_2) \cap B)$, and $N_{G_2}(A_2) \cap B \neq 1$. Let b be an involution in $N_{G_2}(A_2) \cap B$ and let $K = H_2 \cap H_2^b$. Then $A_2, H_{2'}, \{b\} \subseteq N_G(K)$, (where $H_{2'}$ is the 2-complement of H) so $A < N_G(K)$, which, by the maximality of A, implies $N_G(K) = G$. Since G is not solvable, we must have $K = 1$. This gives

$$2\,|H_2| = |A_2| = |H_2 \cdot H_2^b| = |H_2| \cdot |H_2^b|$$

or $|H_2| = 2$, as claimed.

Therefore G_2 must be the dihedral group of order 8.

To see this, we note first that B_2 is a subgroup of G_2 of index 4, and B_2 does not intersect the center of G_2, for if $x \in B_2 \cap Z(G_2)$, $x \neq 1$ then $x \in N_G(H_2)$, contrary to $N_G(H_2) = A$. Now consider the action of G_2 on the right cosets of B_2 in G_2. This gives a homomorphism from G_2 into the symmetric group on 4 letters, which is an injection since the kernel is contained in B_2 and $B_2 \cap Z(G_2) = 1$, and as $|G_2| > 4$, we must have G_2 isomorphic to the Sylow 2-subgroup of the symmetric group on 4 letters, the dihedral group of order 8.

Call $B_2 = \langle x \rangle$ and $H_2 = \langle y \rangle$. Now since $|B_2| = 2$, we can apply Lemma 1 and conclude that G has a subgroup N of index 2 and that $y \notin N$. As in the previous theorem, N is the only proper normal subgroup of G, and so N is the direct product of isomorphic non-Abelian simple groups.

Now since $B \leqslant N_G(B_2)$, this subgroup is solvable, and we may take C to be a Hall $\pi(B)$-subgroup of $N_G(B_2)$ containing B. Then $[C : B] = 2$, and $G = HC$. Since $G = A \cdot N_G(B_2)$, we have $H_2 \nleqslant N_G(B_2)$ by Theorem G, and so $H \cap C = 1$. Thus we may apply Lemma 1 again and conclude that $x \notin N$. But since $[G : N] = 2$, we must have $xy \in N$. As x and y are non-commuting involutions of the dihedral group of order 8, xy has order 4, and since $|N|_2 = 4$, N has a cyclic Sylow 2-subgroup, contary to N being the direct product of non-Abelian simple groups. This contradiction completes the proof of Theorem 3 and so of the Main Theorem.

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